

Fair Spectral Clustering

Spectral Clustering (SC) partitions data $\{x_i\}_{i=1}^n$ by finding k leading eigenvectors $H \in \mathbb{R}^{n \times k}$ of the normalized affinity matrix M :

$$\max_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^\top MH) \quad \text{s.t.} \quad H^\top H = I_k.$$

Fair Spectral Clustering (Fair SC) [1] introduces group fairness:

- h demographic groups $V_1, \dots, V_h$, e.g., age, gender, etc.
- Group membership matrix $G_{is} = 1$ if $x_i \in V_s$, else 0.
- **Group Fairness:** Each cluster C_l should represent groups proportionally to their presence in the overall dataset:

$$\frac{|V_s \cap C_l|}{|C_l|} = \frac{|V_s|}{n}, \quad \forall s \in [h], l \in [k].$$

- This translates to a linear constraint $F^\top H = 0$, where $F = D^{-1/2}(G - \mathbf{1}_n \mathbf{z}^\top)$ encodes group information [1].

The **Fair SC problem** is:

$$\begin{aligned} \max_{H \in \mathbb{R}^{n \times k}} \quad & \text{Tr}(H^\top MH) \\ \text{s.t.} \quad & H^\top H = I_k, \\ & F^\top H = 0. \end{aligned} \quad (1)$$

Existing Methods & Bottlenecks:

- **o-FSC** [1]: Eigendecomposition of $Z^\top MZ$, with Z nullspace of $F^\top$.
- **s-FSC** [2]: Uses projected eigendecomposition $PMPY' = Y'\Lambda$. Faster, but still relies on iterative eigensolvers.

Paper TL;DR

We develop a faster, gradient-based method for FairSC by avoiding full eigendecomposition, using Difference-of-Convex (DC) programming and ADMM.

Our Contributions

1. DC Formulation for Eigenproblems: We reformulate (1) into a Difference of Convex (DC) minimization. To avoid computing $M^{1/2}$, we work with M^2 implicitly by optimizing $\|MH\|_F^2$. Primal DC problem:

$$\min_{H \in \mathbb{R}^{n \times k}} g(H) + h(H) - f(MH),$$

where $f(H) = \frac{1}{2} \|H\|_F^2$, $g(H) = \iota_{\mathcal{S}_n^k}(H)$ (orthonormality, can be relaxed to convex hull) and $h(H) = \iota_{\{0\}}(F^\top H)$ (fairness constraint).

2. Novel Variable Augmentation for ADMM: To handle $F^\top H = 0$ within DC, we introduce a purpose-built variable augmentation:

$$\min_{H, Y \in \mathbb{R}^{n \times k}} g(H) + h(Y) - f(MH) \quad \text{s.t.} \quad MH = Y.$$

The fairness constraint $F^\top Y = 0$ is imposed on $Y = MH$.

DC Dualization & ADMM

The Augmented Lagrangian with penalty $\alpha > 0$ is:

$$\begin{aligned} \mathcal{L}_\alpha(H, Y, P) = & g(H) + h(Y) - f(MH) \\ & + \langle P, MH - Y \rangle + \frac{\alpha}{2} \|MH - Y\|_F^2. \end{aligned}$$

- The H -subproblem at iteration $(i+1)$:

$$H^{(i+1)} = \arg \min_H g(H) - \underbrace{\left(f(MH) - \langle P^{(i)}, MH \rangle - \frac{\alpha}{2} \|MH - Y^{(i)}\|_F^2 \right)}_{\text{convex part } \phi(MH)}$$

This is a DC problem: $\min_H g(H) - \phi(MH)$ assuming $\alpha < 1$.

Proposition (Dual of DC). *Let $\phi : X \mapsto f(X) - \langle P, X \rangle - \frac{\alpha}{2} \|X - Y\|_F^2$ with $\alpha < 1$. Then strong duality holds and the dual is:*

$$\inf_{V \in \mathbb{R}^{n \times k}} \phi^*(V) - g^*(MV).$$

- $\phi^*(V)$ is quadratic and its gradient is easy to compute.
- $g^*(MV) = \text{Tr}(\sqrt{V^\top M^2 V})$ is the nuclear norm. Its gradient involves SVD of $V^\top M^2 V \in \mathbb{R}^{k \times k}$ (cheap for small k).

Computing the Primal Solution. Once the dual solution $\hat{V}$ is computed, one can recover the corresponding primal solution $H^{(i+1)}$ by

$$\max_H \langle \hat{V}, MH \rangle \quad \text{s.t.} \quad H^\top H = I_k.$$

Solved as $H^{(i+1)} = LR^\top$ by the SVD (or Newton-Schulz) of $\hat{V}^\top M$.

- The Y -subproblem at iteration $(i+1)$ is solved in closed-form efficiently by projecting onto the nullspace of $F^\top$.

- **Overall Algorithm**

Algorithm 1 ADMM-type method for Fair SC

Input: Affinity K , group matrix $F \in \mathbb{R}^{n \times h}$, k , $\alpha^{(0)} > 0$

- 1: Compute normalized affinity $M = D^{-1/2} K D^{-1/2}$.
- 2: Initialize $H^{(0)}, Y^{(0)}, P^{(0)}$.
- 3: **for** $i = 0, 1, \dots, T-1$ **do**
- 4: $H^{(i+1)} \leftarrow$ Solve H -subproblem (dual L-BFGS)
- 5: $Y^{(i+1)} \leftarrow$ Solve Y -subproblem (nullspace proj.)
- 6: $P^{(i+1)} \leftarrow P^{(i)} + \alpha^{(i)}(MH^{(i+1)} - Y^{(i+1)})$
- 7: Update $\alpha^{(i+1)}$
- 8: **end for**
- 9: Apply k -means on rows of $D^{-1/2} H^{(T)}$.

- **Complexity:** We avoid $n \times n$ eigendecomposition.
 $\Rightarrow$ Dominated by matrix multiplications (applying M to a $n \times k$ matrix): $\mathcal{O}(n^2 k)$.
 $\Rightarrow$ We gain substantial practical speedups.

Experimental Results: Speed & Fairness

Datasets: LastFM, Thyroid, Census, 4area. **Group types:** Nationality, Diseases, Demographic, Research Fields. **Metrics:** Runtime (s), Balance (defined as $\text{balance}(C_l) = \min_{s \neq s'} \frac{|V_s \cap C_l|}{|V_{s'} \cap C_l|}$, closer to 1 is better). **Baselines:** o-FSC [1], s-FSC [2].

- **Real-world Data Runtimes** ($k = 25$)

Dataset	n	o-FSC	s-FSC	Ours
LastFM	5576	103.82	19.08	4.59
Thyroid	7200	279.03	30.49	7.38
Census	32561	-	136.60	15.78
4area	35385	-	166.92	25.85

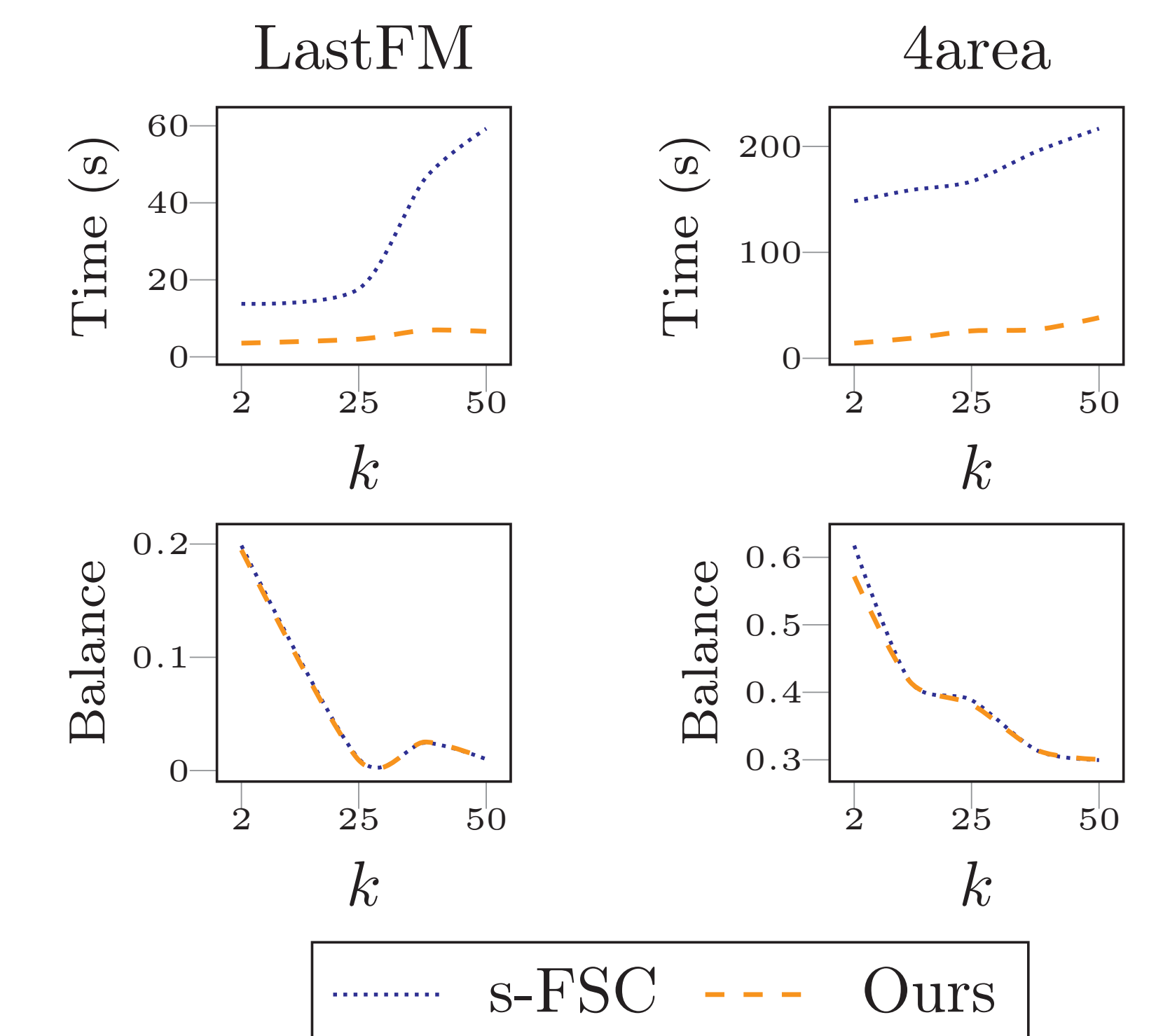
$\Rightarrow$ Our method is consistently faster, especially on larger datasets.

- **Comparison vs. Exact Algorithm**

Dataset	Method	Clustering	Balance	Time
LastFM	s-FSC	$1.057 \pm .063$	$0.0105 \pm .0020$	$4.16 \times$
	Ours	$1.086 \pm .074$	$0.0093 \pm .0015$	$1 \times$
Thyroid	s-FSC	$0.353 \pm .020$	$0.0029 \pm .0005$	$4.13 \times$
	Ours	$0.353 \pm .018$	$0.0030 \pm .0004$	$1 \times$
Census	s-FSC	134.539 ± 3.667	$0.0004 \pm .0001$	$8.65 \times$
	Ours	130.973 ± 14.371	$0.0004 \pm .0001$	$1 \times$
4area	s-FSC	235.268 ± 6.084	$0.3884 \pm .0150$	$6.45 \times$
	Ours	242.000 ± 10.303	$0.3823 \pm .0200$	$1 \times$

$\Rightarrow$ Our method produces comparable clustering assignments and balance to exact methods, while being significantly faster.

- **Runtime and Fairness vs. Number of Clusters k**



$\Rightarrow$ Our method shows even better efficiency gains as k increases. We maintain the fairness quality (balance) of the exact algorithm.

References

- [1] Kleindessner et al. Guarantees for Spectral Clustering with Fairness Constraints. In *ICML* 2019.
- [2] Wang et al. Scalable Spectral Clustering with Group Fairness Constraints. In *AISTATS* 2023.

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